Trigonometry Workbook Solutions

Trig identities

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EVEN-ODD IDENTITIES

1. Find trig function has the largest value at $\theta = -\pi/3$?

$$\sin\left(-\frac{\pi}{3}\right) \qquad \cot\left(-\frac{\pi}{3}\right)$$

$$\cos\left(-\frac{\pi}{3}\right) \qquad \sec\left(-\frac{\pi}{3}\right)$$

$$\tan\left(-\frac{\pi}{3}\right) \qquad \csc\left(-\frac{\pi}{3}\right)$$

Solution:

The angle $\theta = -\pi/3$ lies in quadrant IV. Only the cosine and secant functions of an angle in quadrant IV are positive. So we only need to determine which of those is larger than the other.

Use even-odd identities to get rid of the negative angles.

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

Then

$$\cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

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$$\sec\left(-\frac{\pi}{3}\right) = \sec\frac{\pi}{3} = 2$$

So out of all the trig functions, the secant function has the largest value at $\theta = -\pi/3$.

2. Evaluate the expression.

$$\sec(-45^\circ) \cdot \csc(-45^\circ) \cdot \tan(-45^\circ) \cdot \cot(-45^\circ)$$

Solution:

Use even-odd identities to get rid of the negative angles.

$$\tan(-\theta) = -\tan\theta$$

$$\cot(-\theta) = -\cot\theta$$

$$\sec(-\theta) = \sec\theta$$

$$\csc(-\theta) = -\csc\theta$$

Sot the expression becomes

$$\sec(-45^\circ) \cdot \csc(-45^\circ) \cdot \tan(-45^\circ) \cdot \cot(-45^\circ)$$

$$\sec 45^\circ \cdot (-\csc 45^\circ) \cdot (-\tan 45^\circ) \cdot (-\cot 45^\circ)$$

$$\sqrt{2} \cdot \left(-\sqrt{2}\right) \cdot (-1) \cdot (-1)$$

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$$-2 \cdot 1$$

$$-2$$

■ 3. Solve the equation and write the solution in set notation.

$$\sin(-x) + \cos(-x) = 0$$

Solution:

Use even-odd identities to get rid of the negative angles.

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

Then

$$-\sin x + \cos x = 0$$

$$\sin x = \cos x$$

$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

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These are the solutions in $[0,2\pi)$, but if we extend the solutions into other periods, we get

$x = \frac{\pi}{4} + \pi k$, where k is an integer

$$\left\{ \frac{\pi}{4} + \pi k : \ k \in \mathbb{Z} \right\}$$

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SUM-DIFFERENCE IDENTITIES

1. Evaluate the expression.

$$\cos\frac{7\pi}{12}$$

Solution:

The angle $\theta = 7\pi/12$ isn't on the standard unit circle, so we may not know its cosine value. But we can write the angle as the sum of two other angles, $\theta = 4\pi/12$ and $\theta = 3\pi/12$.

$$\cos\frac{7\pi}{12} = \cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right) = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$

Then we can apply the sum identity

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

to rewrite the trig function as

$$\cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$\frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$

$$\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$

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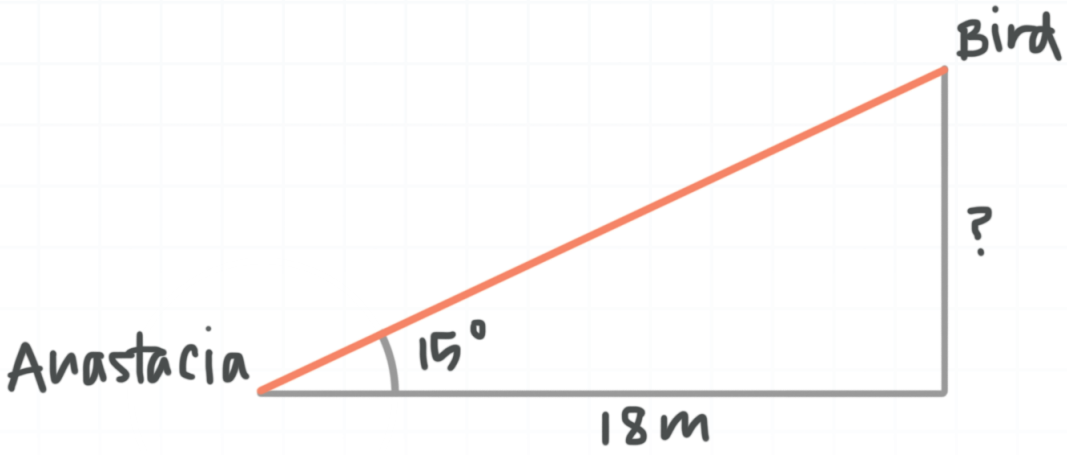
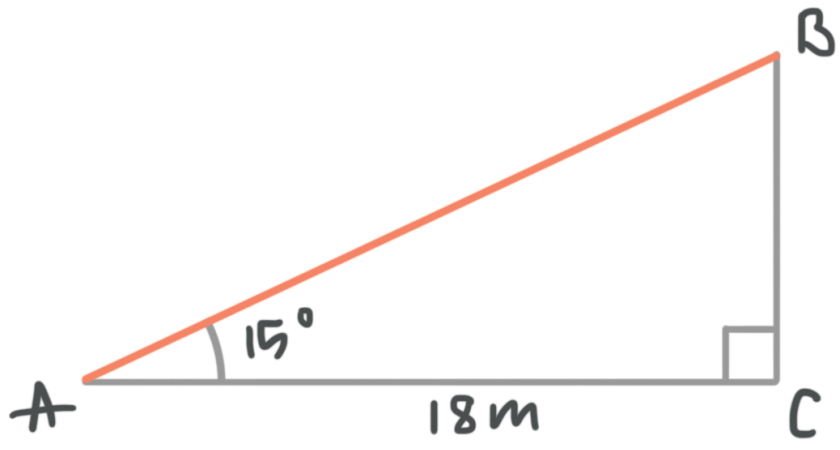
$$\frac{\sqrt{2}-\sqrt{6}}{4}$$

2. Anastacia is watching a bird on a tree. She measured the angle of elevation of the bird as $15°$, and the distance to the tree as 18 meters. Find the exact altitude of the bird above the ground.



Solution:

Call the right triangle ABC where A is Anastacia's position, B is the bird's position, and C is the point on the ground just below the bird.



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BC is the side we're interested in, and by the definition of tangent, we can say

$$\tan A = \frac{BC}{AC}$$

$$BC = AC \cdot \tan A$$

$$BC = 18 \tan 15^\circ$$

To find the exact value of $\tan 15^\circ$, rewrite the angle as the difference $45^\circ - 30^\circ$.

$$BC = 18 \tan(45^\circ - 30^\circ)$$

Then apply the sum identity

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta \tan\phi}$$

to get

$$BC = 18 \cdot \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$

$$BC = 18 \cdot \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}}$$

$$BC = 18 \cdot \frac{3 - \sqrt{3}}{3 + \sqrt{3}}$$

Rationalize the denominator.

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$$BC = 18 \cdot \frac{\left(3 - \sqrt{3}\right)^2}{\left(3 + \sqrt{3}\right)\left(3 - \sqrt{3}\right)}$$

$$BC = 18 \cdot \frac{9 - 6\sqrt{3} + 3}{9 - 3}$$

$$BC = 18 \cdot \frac{12 - 6\sqrt{3}}{6}$$

$$BC = 18\left(2 - \sqrt{3}\right)$$

$$BC = 36 - 18\sqrt{3} \text{ m}$$

3. Solve the equation.

$$\sin x + \sqrt{3}\cos x = \sqrt{2}$$

Solution:

When you see an equation with a linear combination of sine and cosine, the best way to solve it is using this trick with a sum-difference identity.

Divide both sides of the equation by 2.

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{2}}{2}$$

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We know that

$$\sin 30^\circ = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

Which means we can replace the fractions like this:

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{2}}{2}$$

$$\sin 30^\circ \sin x + \cos 30^\circ \cos x = \frac{\sqrt{2}}{2}$$

$$\cos 30^\circ \cos x + \sin 30^\circ \sin x = \frac{\sqrt{2}}{2}$$

Now apply the difference identity

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

in the reverse order, with $\theta = x$ and $\phi = 30^\circ$.

$$\cos 30^\circ \cos x + \sin 30^\circ \sin x = \cos(x - 30^\circ)$$

So

$$\cos(x - 30^\circ) = \frac{\sqrt{2}}{2}$$

If the cosine function is equivalent to $\sqrt{2}/2$, then it has two solutions in $[0,2\pi)$, which are 45° and $360^\circ - 45^\circ = 315^\circ$. So

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$x_1 - 30^\circ = 45^\circ + 360^\circ n$, where n is any integer

$x_1 = 75^\circ + 360^\circ n$, where n is any integer

and

$x_2 - 30^\circ = 315^\circ + 360^\circ m$, where m is any integer

$x_2 = 345^\circ + 360^\circ m$, where m is any integer

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HALF-ANGLE IDENTITIES

■ 1. Use half-angle identities to find the exact value.

$$\tan 67.5^\circ$$

Solution:

Rewrite the angle as

$$67.5^\circ = \frac{135^\circ}{2} = \frac{90^\circ + 45^\circ}{2}$$

Use the half-angle identities

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Divide the sine identity by the cosine identity.

$$\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\pm\sqrt{\frac{1-\cos\theta}{2}}}{\pm\sqrt{\frac{1+\cos\theta}{2}}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

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Then plug in $\theta = 135°$ or $\theta/2 = 67.5°$.

$$\tan 67.5° = \sqrt{\frac{1 - \cos 135°}{1 + \cos 135°}}$$

$$\tan 67.5° = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{1 - \frac{\sqrt{2}}{2}}} = \sqrt{\frac{\frac{2 + \sqrt{2}}{2}}{\frac{2 - \sqrt{2}}{2}}} = \sqrt{\frac{2 + \sqrt{2}}{2 - \sqrt{2}}} = \sqrt{\frac{\left(2 + \sqrt{2}\right)^2}{\left(2 - \sqrt{2}\right)\left(2 + \sqrt{2}\right)}}$$

$$\tan 67.5° = \sqrt{\frac{\left(2 + \sqrt{2}\right)^2}{2^2 - \left(\sqrt{2}\right)^2}} = \frac{2 + \sqrt{2}}{\sqrt{4 - 2}} = \frac{2 + \sqrt{2}}{\sqrt{2}} = \sqrt{2} + 1$$

■ 2. A triangular-shaped plot of land has a side 200 meters long that's adjacent to a river. The angles adjacent to that side are $\alpha = \pi/4$ and $\beta = 3\pi/8$. Find the exact length of a fence that divides the plot into two equal parts and bisects α.

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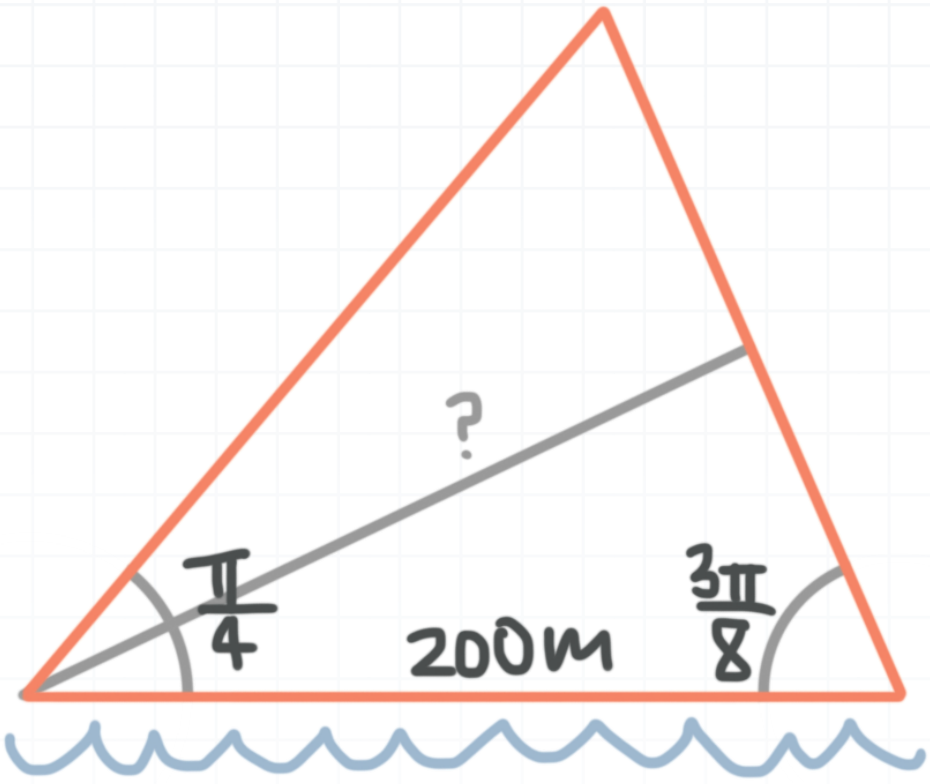
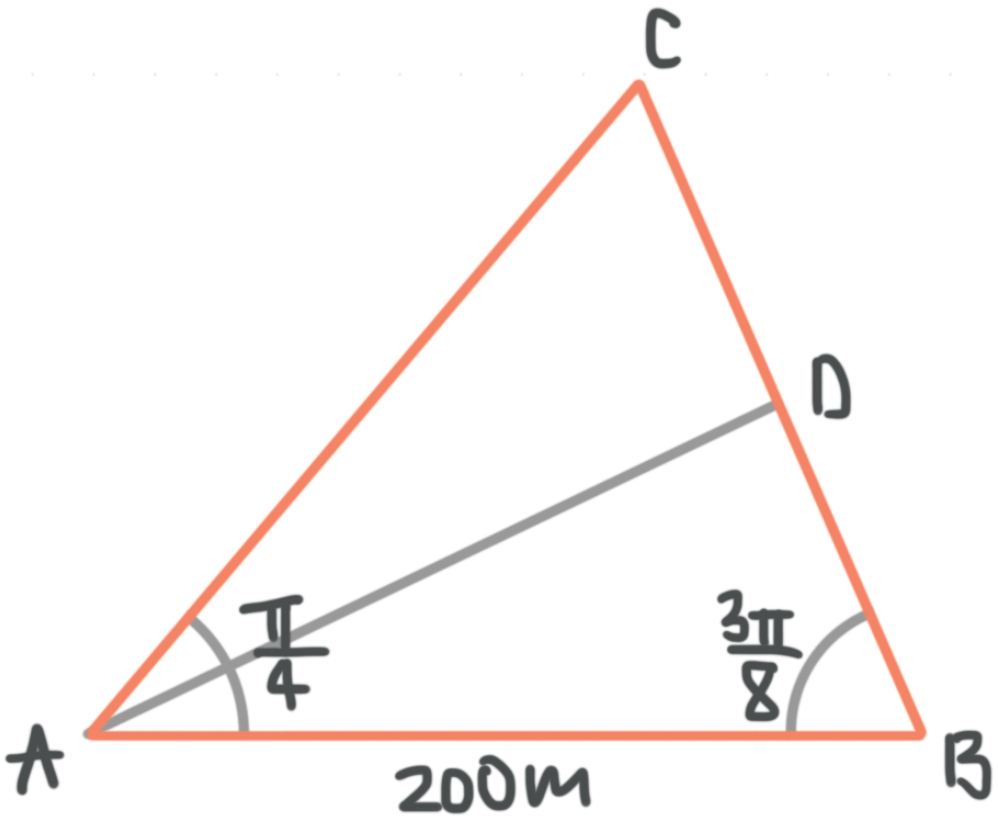


Solution:

If the triangle is ABC, α is at A, β is at B, and AD is the bisector of α, then AD is the length we want to find.



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The sum of the angles in any triangle is π, so the measure of angle ADB is

$$m\angle ADB = \pi - \beta - \frac{\alpha}{2}$$

$$m\angle ADB = \pi - \frac{3\pi}{8} - \frac{\pi}{8}$$

$$m\angle ADB = \frac{\pi}{2}$$

So the triangle ABD is a right triangle. We know the hypotenuse $AB = 200$ m and $m\angle DAB = \pi/8$. We just need to find the leg AD. Then

$$\cos DAB = \frac{AD}{AB}$$

$$AD = AB \cdot \cos DAB$$

$$AD = 200 \cos \frac{\pi}{8}$$

Use the half-angle identity

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

Since $\pi/8$ lies in the first quadrant, its cosine is positive, so

$$\cos \frac{\pi}{8} = \sqrt{\frac{1 + \cos \frac{\pi}{4}}{2}} = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2 + \sqrt{2}}{4}} = \frac{\sqrt{2 + \sqrt{2}}}{2}$$

So the length of AD is

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$$AD = 200\cos\frac{\pi}{8}$$

$$AD = 200 \cdot \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$AD = 100\sqrt{2+\sqrt{2}}$$

■ 3. Simplify the expression for the angle t that lies in the second quadrant.

$$\sin\frac{t}{2}\cos\frac{t}{2}$$

Solution:

Use the half-angle identities

$$\sin\frac{t}{2} = \pm\sqrt{\frac{1-\cos t}{2}}$$

$$\cos\frac{t}{2} = \pm\sqrt{\frac{1+\cos t}{2}}$$

Since t lies in the second quadrant, $t/2$ is in the first quadrant, and its sine and cosine values are positive.

$$\sin\frac{t}{2} = \sqrt{\frac{1-\cos t}{2}}$$

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$$\cos\frac{t}{2}=\sqrt{\frac{1+\cos t}{2}}$$

Substitute them into the expression.

$$\sin\frac{t}{2}\cos\frac{t}{2}$$

$$\sqrt{\frac{1-\cos t}{2}}\sqrt{\frac{1+\cos t}{2}}$$

$$\sqrt{\frac{(1+\cos t)(1-\cos t)}{4}}$$

$$\sqrt{\frac{1-\cos^2 t}{4}}$$

$$\sqrt{\frac{\sin^2 t}{4}}$$

Since t is in the second quadrant, its sine is positive, so the expression can be simplified to

$$\frac{\sin t}{2}$$

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DOUBLE-ANGLE IDENTITIES

■ 1. Simplify the expression.

$$\cos^4\theta - \sin^4\theta$$

Solution:

Rewrite the expression as the difference of squares.

$$\cos^4\theta - \sin^4\theta$$

$$(\cos^2\theta)^2 - (\sin^2\theta)^2$$

Because it's the difference of squares, use the algebraic identity $a^2 - b^2 = (a - b)(a + b)$.

$$(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$$

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$,

$$(\cos^2\theta - \sin^2\theta) \cdot 1$$

$$\cos^2\theta - \sin^2\theta$$

Use the double-angle identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

$$\cos 2\theta$$

So

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$$\cos^4\theta - \sin^4\theta = \cos 2\theta$$

■ 2. Find the exact value of the expression.

$$4\cos 75^\circ \cos 15^\circ$$

Solution:

Use a transformation for $\cos 75^\circ$.

$$\cos 75^\circ = \cos(90^\circ - 15^\circ)$$

Then use a difference identity on the right side.

$$\cos(90^\circ - x) = \cos 90^\circ \cos x + \sin 90^\circ \sin x$$

$$\cos(90^\circ - x) = 0 \cdot \cos x + 1 \cdot \sin x$$

$$\cos(90^\circ - x) = \sin x$$

So

$$4\cos 75^\circ \cos 15^\circ = 4\sin 15^\circ \cos 15^\circ$$

Then use the double-angle identity

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

to rewrite it as

$$4\sin 15^\circ \cos 15^\circ$$

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$$2 \cdot 2 \sin 15^\circ \cos 15^\circ$$

$$2 \sin 30^\circ$$

$$2 \cdot \frac{1}{2}$$

$$1$$

3. Sketch the graph of $f(x)$.

$$f(x) = 4 \sin x \sin (90^\circ - x) + 1$$

Solution:

To transform $\sin(90^\circ - x)$, use the difference identity

$$\sin(90^\circ - x) = \sin 90^\circ \cos x - \cos 90^\circ \sin x$$

$$\sin(90^\circ - x) = 1 \cdot \cos x - 0 \cdot \sin x$$

$$\sin(90^\circ - x) = \cos x$$

After simplifying, the function is

$$f(x) = 4 \sin x \cos x + 1$$

Then use the double-angle identity

$$\sin(2x) = 2 \sin x \cos x$$

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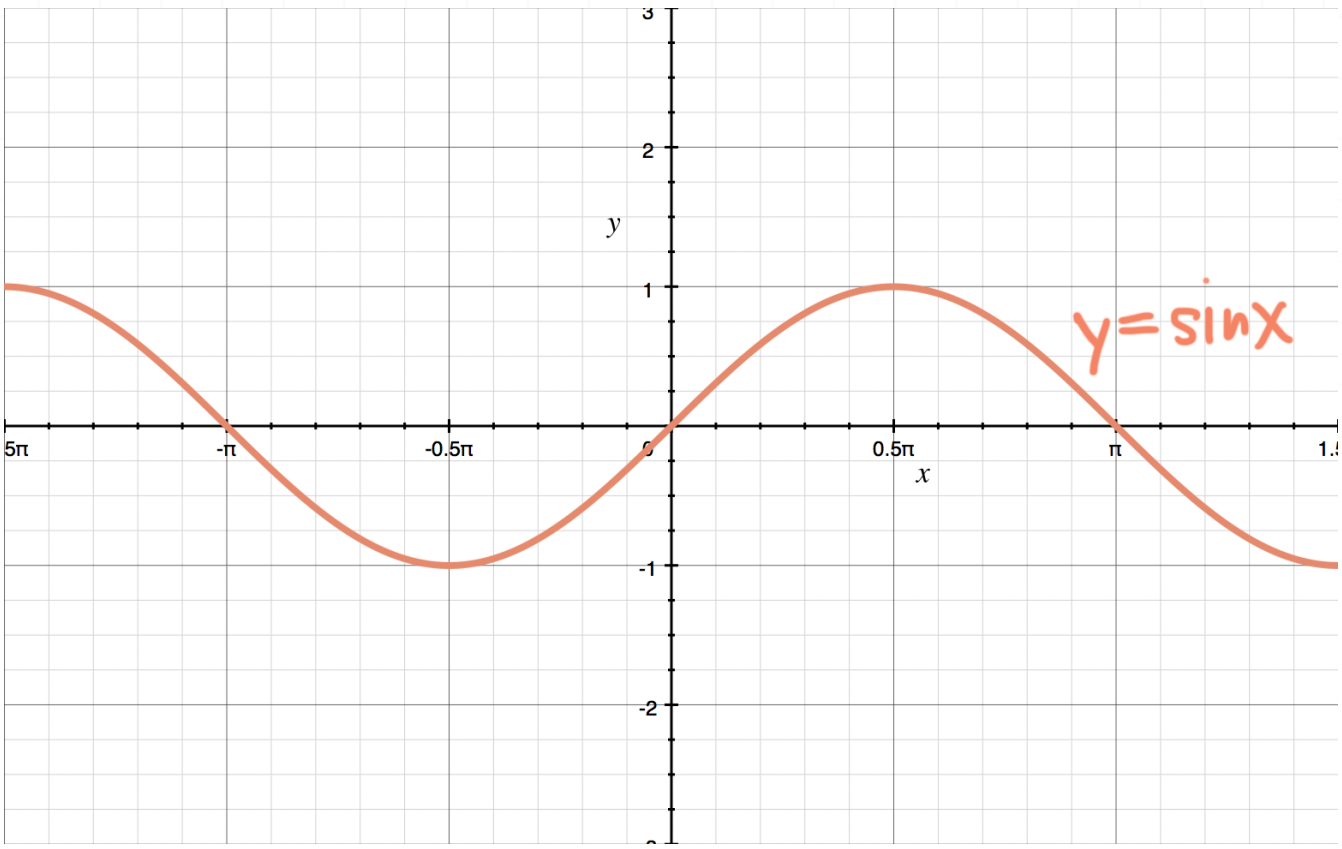
to get

$$f(x) = 2 \cdot 2 \sin x \cos x + 1$$

$$f(x) = 2 \sin(2x) + 1$$

Now we can easily sketch the graph, because we know how to sketch $\sin x$, and all we have to do is to apply a sequence of transformations to it. Start with $g(x) = \sin x$.



Compress horizontally by 2 to get $g_1(x) = g(2x) = \sin(2x)$.

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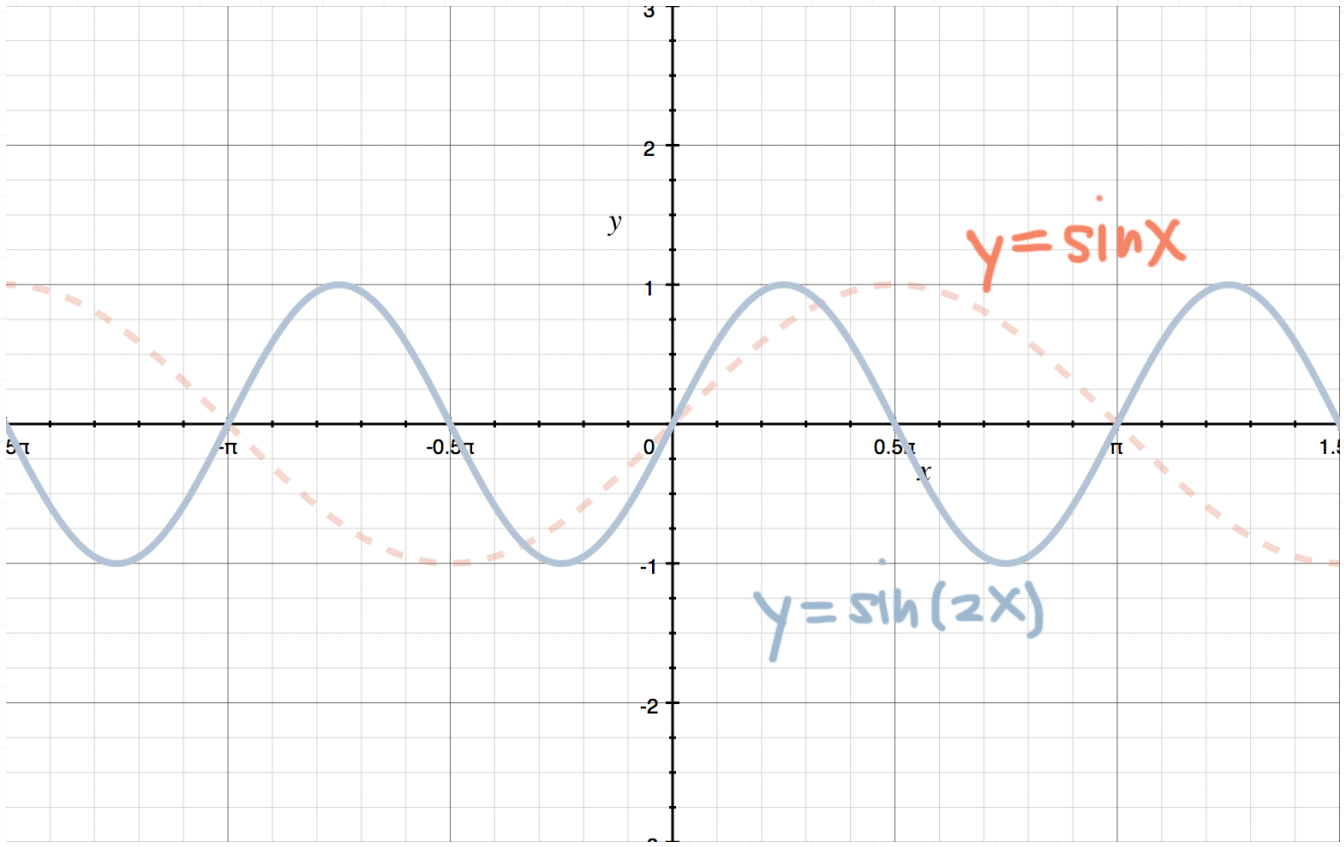
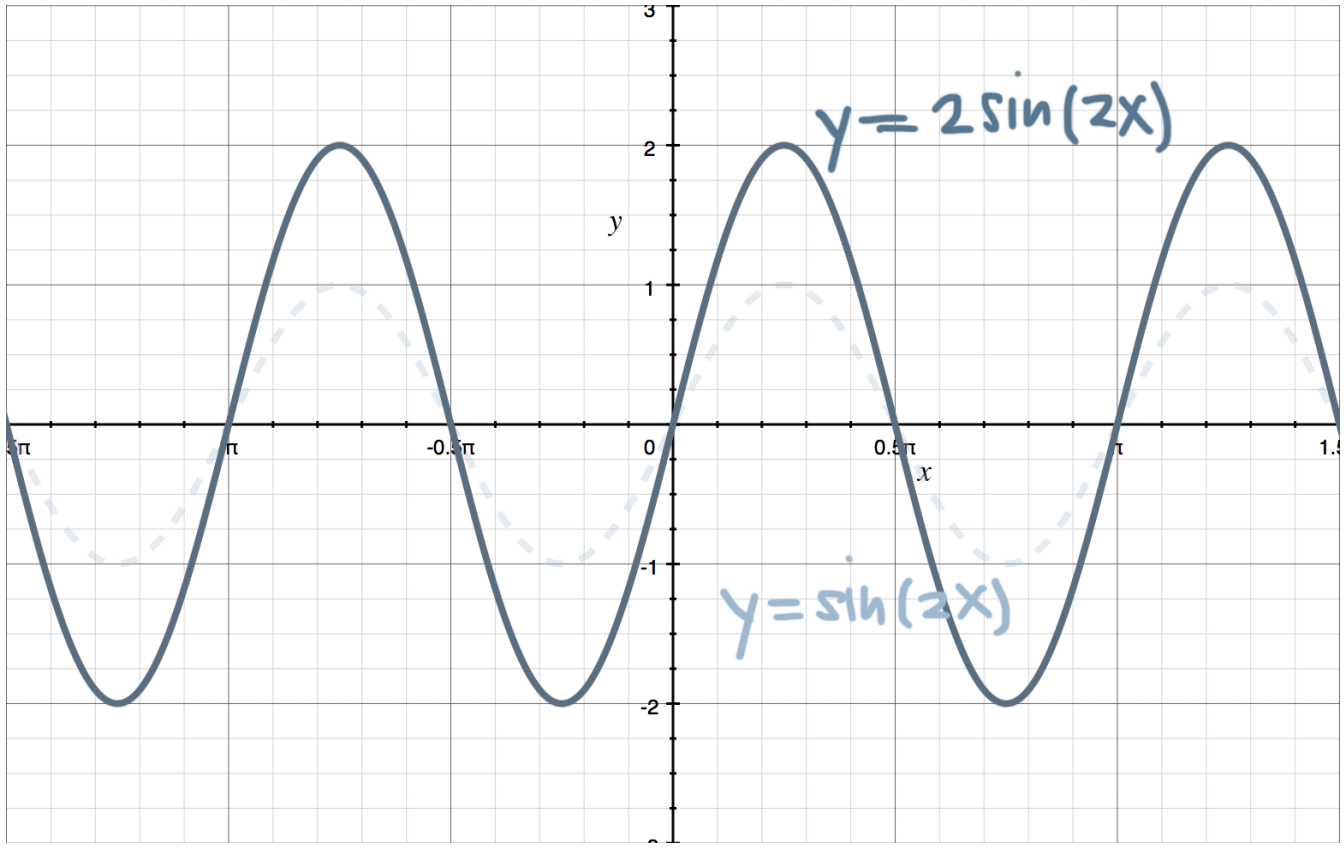


Stretch vertically by 2 to get $g_2(x) = 2g_1(x) = 2\sin(2x)$.



Shift upward by 1 to get $g_3(x) = g_2(x) + 1 = 2\sin(2x) + 1$.

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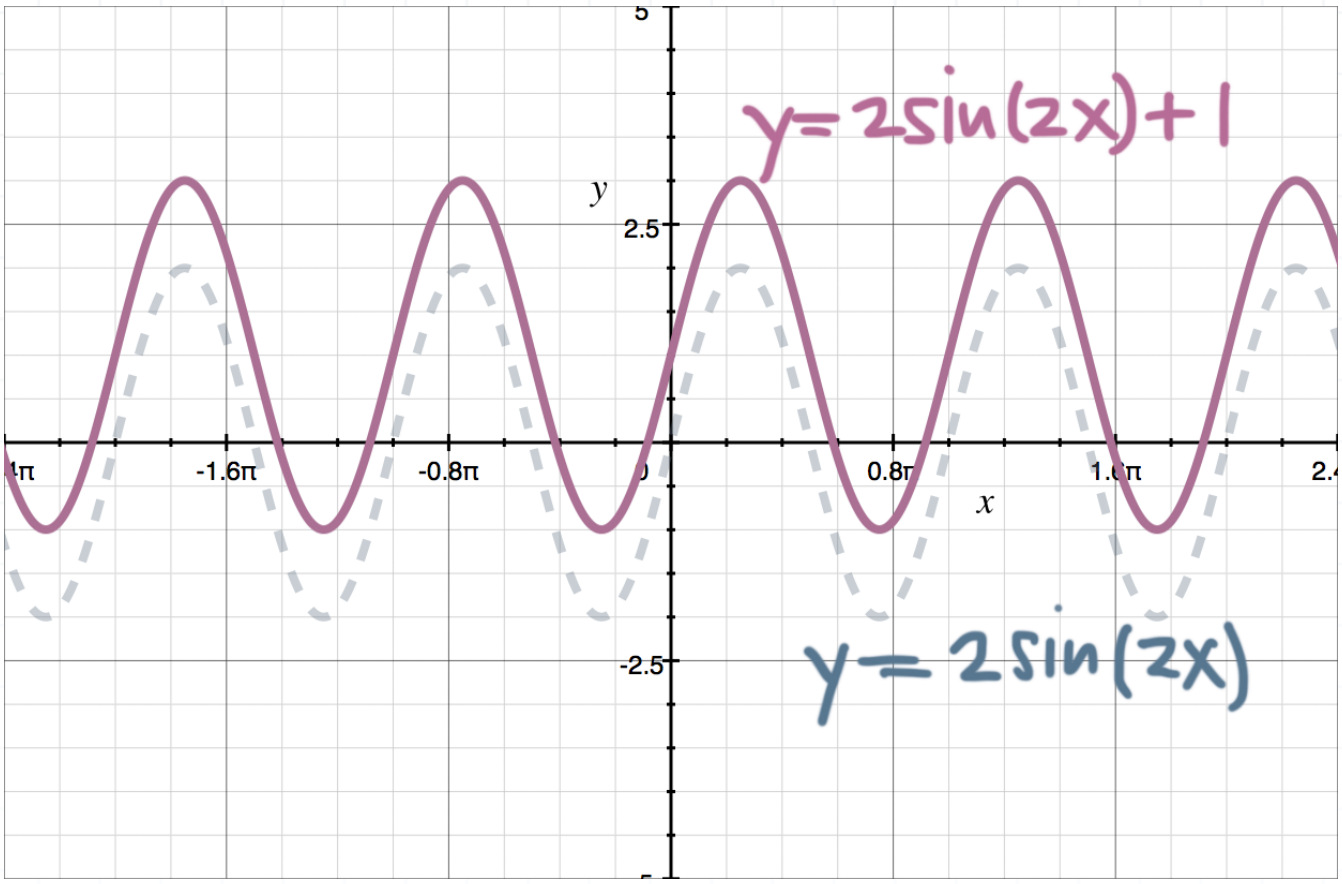

So the graph of $f(x) = 2\sin(2x) + 1$ is

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PRODUCT-TO-SUM IDENTITIES

■ 1. Rewrite $\sin^3 x$ as an expression in terms of only first-degree sine functions.

Solution:

First, split up the factors.

$$\sin^3 x = \sin x \cdot \sin x \cdot \sin x$$

For two of the three factors, $\sin x \cdot \sin x$, apply the product-to-sum identity,

$$\sin\theta \sin\phi = \frac{1}{2}\left(\cos(\theta - \phi) - \cos(\theta + \phi)\right)$$

to get

$$\sin^3 x = \left(\frac{1}{2}\left(\cos(x - x) - \cos(x + x)\right)\right) \sin x$$

$$\sin^3 x = \left(\frac{1}{2}(\cos 0 - \cos 2x)\right) \sin x$$

$$\sin^3 x = \left(\frac{1}{2}(1 - \cos 2x)\right) \sin x$$

$$\sin^3 x = \frac{1}{2}(1 - \cos 2x) \sin x$$

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$$\sin^3 x = \frac{1}{2}\left(\sin x - \sin x \cos 2x\right)$$

$$\sin^3 x = \frac{1}{2}\sin x - \frac{1}{2}\sin x \cos 2x$$

Apply the product-to-sum identity,

$$\cos\theta \sin\phi = \frac{1}{2}\left(\sin(\theta+\phi) - \sin(\theta-\phi)\right)$$

to get

$$\sin^3 x = \frac{1}{2}\sin x - \frac{1}{2}\left(\frac{1}{2}\left(\sin(2x+x) - \sin(2x-x)\right)\right)$$

$$\sin^3 x = \frac{1}{2}\sin x - \frac{1}{4}\left(\sin(2x+x) - \sin(2x-x)\right)$$

$$\sin^3 x = \frac{1}{2}\sin x - \frac{1}{4}\sin(2x+x) + \frac{1}{4}\sin(2x-x)$$

$$\sin^3 x = \frac{1}{2}\sin x - \frac{1}{4}\sin(3x) + \frac{1}{4}\sin x$$

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$$

2. Sketch the graph of the function $f(x)$.

$$f(x) = 4\cos\left(x - \frac{\pi}{4}\right)\cos\left(x + \frac{\pi}{4}\right)$$

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Solution:

First apply the product-to-sum identity

$$\cos\theta\cos\phi = \frac{1}{2}\left(\cos(\theta+\phi)+\cos(\theta-\phi)\right)$$

to get

$$f(x) = 4\cos\left(x-\frac{\pi}{4}\right)\cos\left(x+\frac{\pi}{4}\right)$$

$$f(x) = 4\left(\frac{1}{2}\left(\cos\left(x-\frac{\pi}{4}+x+\frac{\pi}{4}\right)+\cos\left(x-\frac{\pi}{4}-x-\frac{\pi}{4}\right)\right)\right)$$

$$f(x) = 2\left(\cos(2x)+\cos\left(-\frac{\pi}{2}\right)\right)$$

Use the even-odd identity $\cos(-\theta) = \cos\theta$ to get

$$f(x) = 2\left(\cos(2x)+\cos\frac{\pi}{2}\right)$$

$$f(x) = 2\left(\cos(2x)+0\right)$$

$$f(x) = 2\cos(2x)$$

Now we can easily sketch the graph as a sequence of transformations, starting with $g(x) = \cos(x)$ and working toward $f(x) = 2\cos(2x)$. The function $g(x) = \cos(x)$ is

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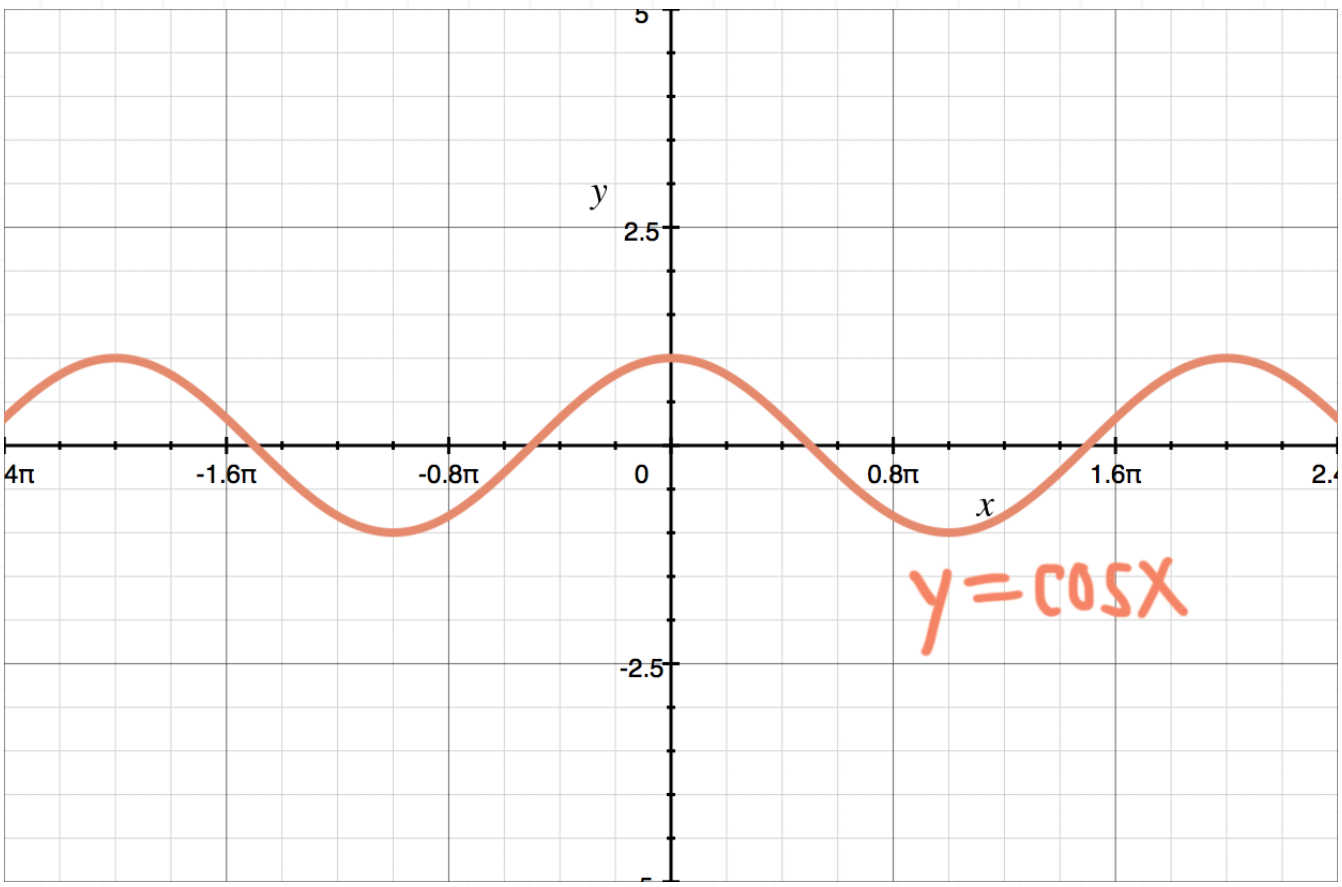
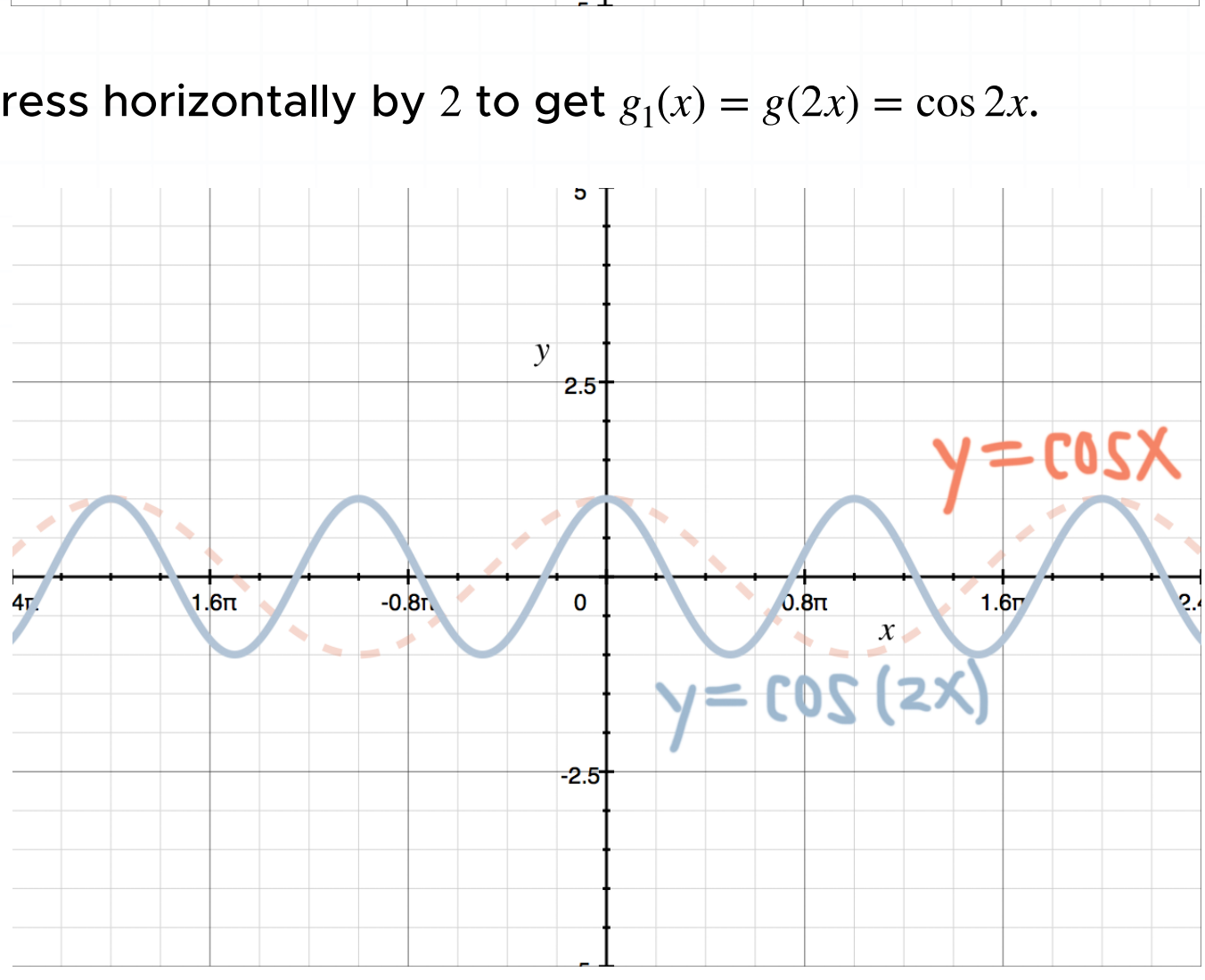


Compress horizontally by 2 to get $g_1(x) = g(2x) = \cos 2x$.



Stretch vertically by 2 to get $g_2(x) = 2g_1(x) = 2\cos 2x$.

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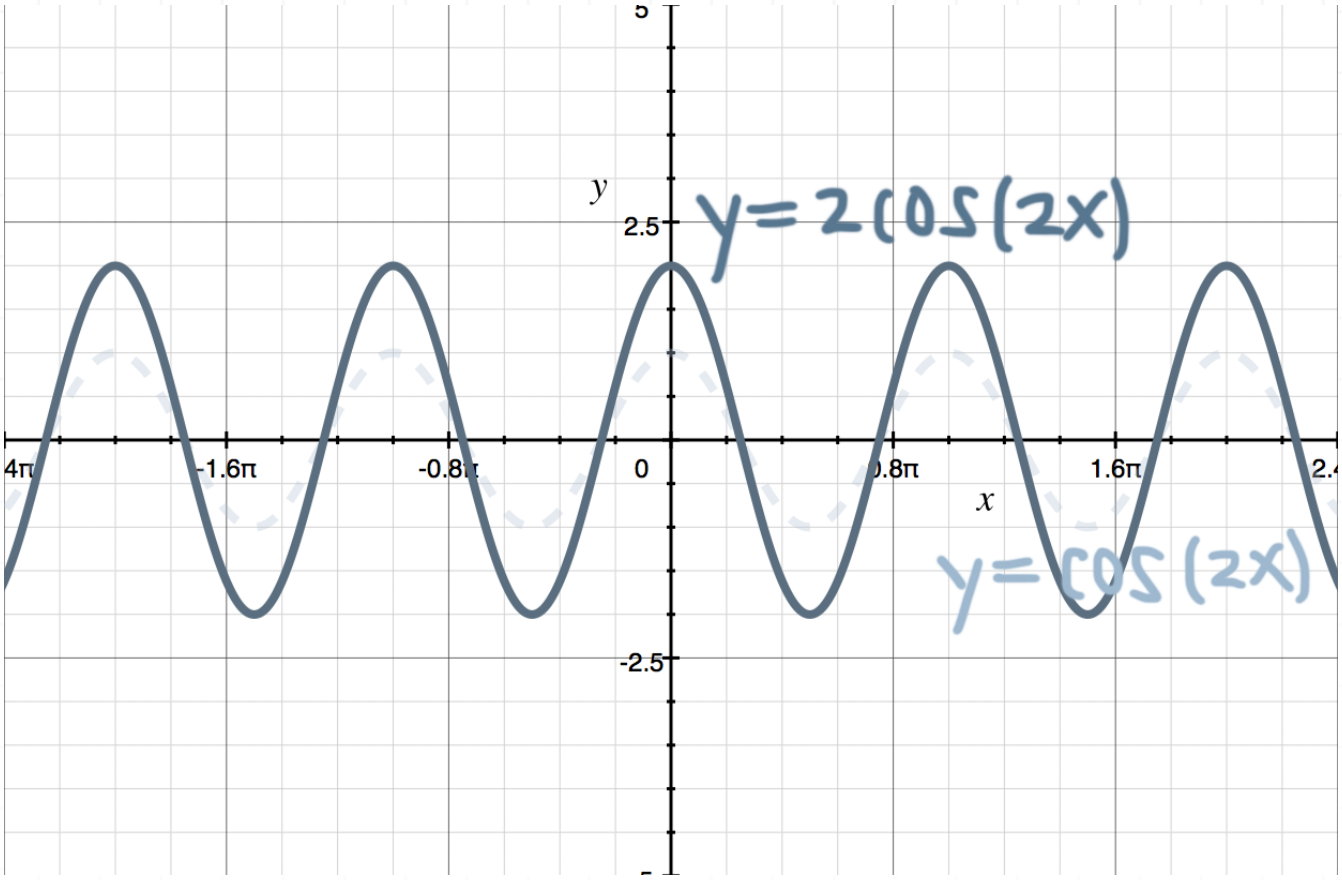
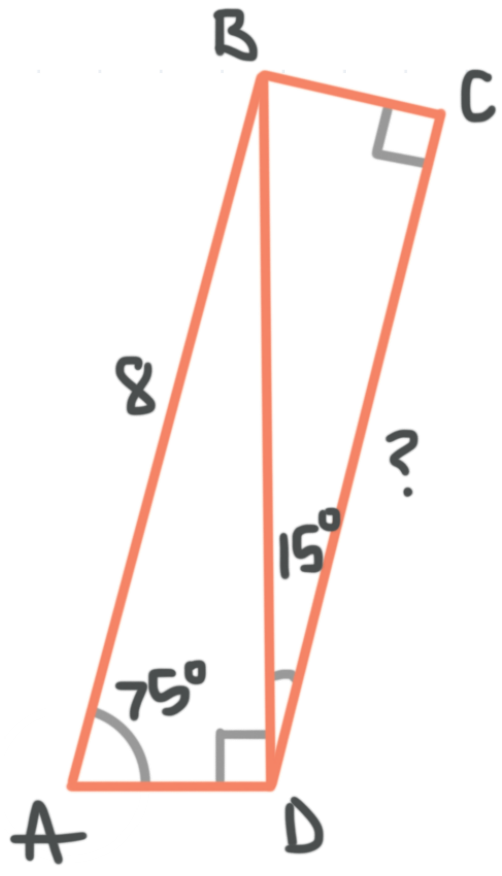


So this dark blue curve is the sketch of $f(x) = 2\cos(2x)$.

3. In quadrilateral $ABCD$, find the exact length of CD.



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Solution:

First we'll find BD in triangle ABD, and then we'll find CD in triangle BCD.

In right triangle ABD,

$$\sin A = \frac{BD}{AB}$$

$$BD = AB \cdot \sin A$$

$$BD = 8 \sin 75^\circ$$

Then in right triangle BCD, we can say

$$\cos BDC = \frac{CD}{BD}$$

$$CD = BD \cdot \cos BDC$$

$$CD = 8 \sin 75^\circ \cos 15^\circ$$

To calculate the exact value of CD, apply the product-to-sum identity

$$\sin\theta \cos\phi = \frac{1}{2}\left(\sin(\theta + \phi) + \sin(\theta - \phi)\right)$$

to get

$$CD = 8\left(\frac{1}{2}\left(\sin(75^\circ + 15^\circ) + \sin(75^\circ - 15^\circ)\right)\right)$$

$$CD = 4\left(\sin(90^\circ) + \sin(60^\circ)\right)$$

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$$CD = 4\left(1 + \frac{\sqrt{3}}{2}\right)$$

$$CD = 4 + 2\sqrt{3}$$

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SUM-TO-PRODUCT IDENTITIES

■ 1. Sketch the graph of $f(x)$.

$$f(x) = \sin\left(x + \frac{\pi}{6}\right) - \sin\left(x - \frac{\pi}{6}\right)$$

Solution:

Apply the difference-to-product identity

$$\sin\theta - \sin\phi = 2\cos\frac{\theta + \phi}{2}\sin\frac{\theta - \phi}{2}$$

to the function to get

$$f(x) = \sin\left(x + \frac{\pi}{6}\right) - \sin\left(x - \frac{\pi}{6}\right)$$

$$f(x) = 2\cos\frac{x + \frac{\pi}{6} + x - \frac{\pi}{6}}{2}\sin\frac{x + \frac{\pi}{6} - \left(x - \frac{\pi}{6}\right)}{2}$$

$$f(x) = 2\cos\frac{2x}{2} \cdot \sin\frac{\frac{\pi}{3}}{2}$$

$$f(x) = 2\cos x \cdot \sin\frac{\pi}{6}$$

$$f(x) = 2\cos x \cdot \frac{1}{2}$$

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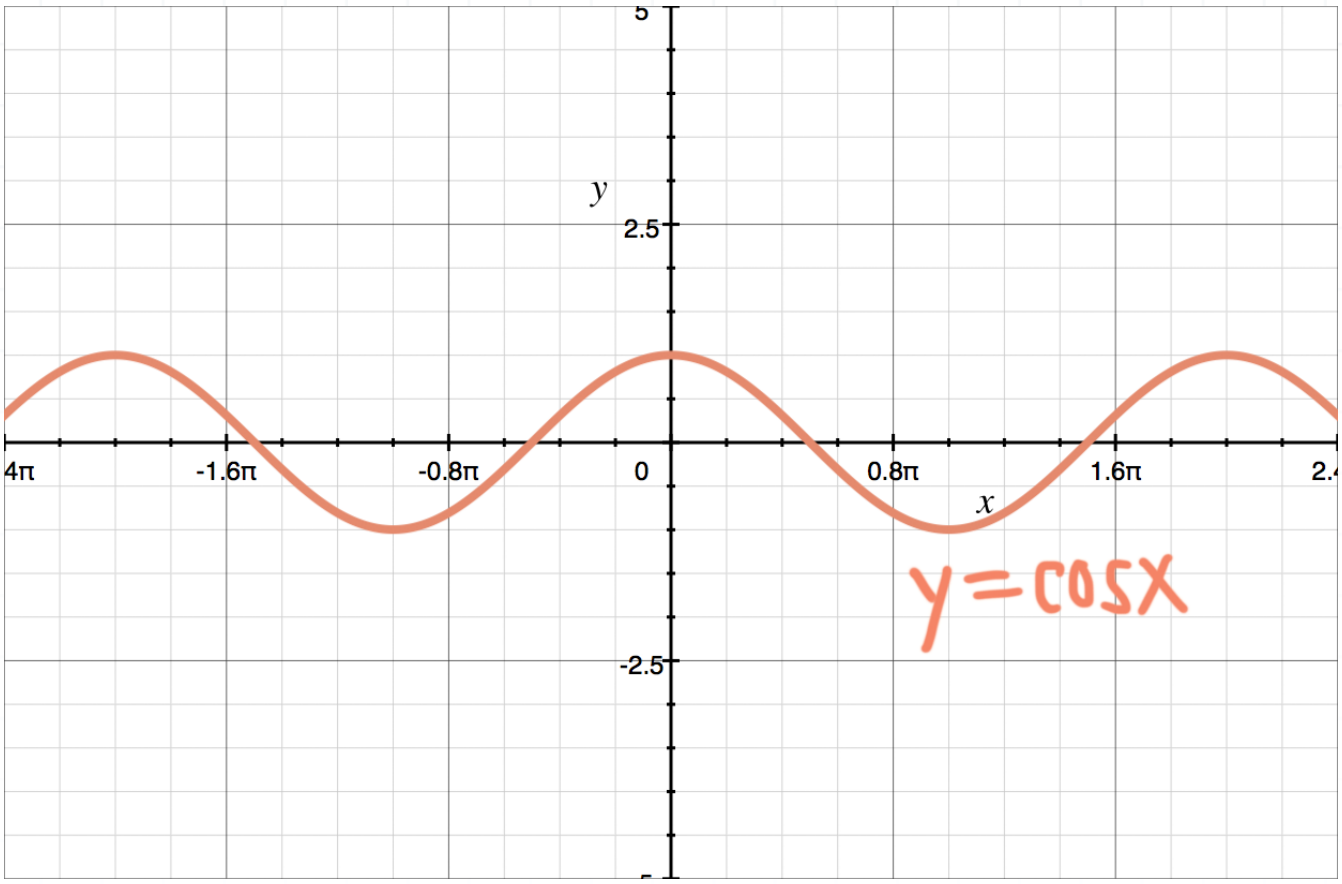
$$f(x) = \cos x$$

The function $f(x)$ is just the standard cosine function $f(x) = \cos x$, so a sketch of its graph is



■ 2. Solve the equation.

$$\cos\left(x + \frac{3\pi}{4}\right) = \cos\left(\frac{3\pi}{4} - x\right)$$

Solution:

Move all terms to the left side of the equation.

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$$\cos\left(x+\frac{3\pi}{4}\right)-\cos\left(\frac{3\pi}{4}-x\right)=0$$

Apply the difference-to-product identity

$$\cos\theta-\cos\phi=-2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

to get

$$\cos\left(x+\frac{3\pi}{4}\right)-\cos\left(\frac{3\pi}{4}-x\right)=0$$

$$-2\sin\frac{x+\frac{3\pi}{4}+\frac{3\pi}{4}-x}{2}\sin\frac{x+\frac{3\pi}{4}-\left(\frac{3\pi}{4}-x\right)}{2}=0$$

$$-2\sin\frac{2\cdot\frac{3\pi}{4}}{2}\cdot\sin\frac{2x}{2}=0$$

$$-2\sin\frac{3\pi}{4}\cdot\sin x=0$$

$$-2\cdot\frac{\sqrt{2}}{2}\cdot\sin x=0$$

$$-\sqrt{2}\sin x=0$$

$$\sin x=0$$

This is true at $-\pi$, 0, π, 2π, $\ldots$, which is really just $x=\pi k$, where k is any integer.

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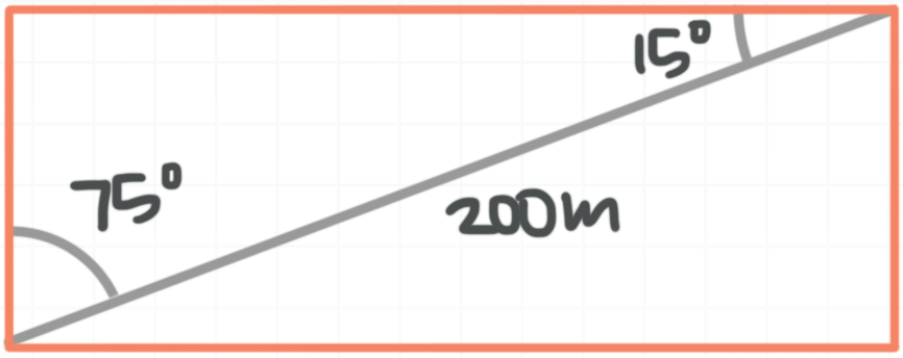
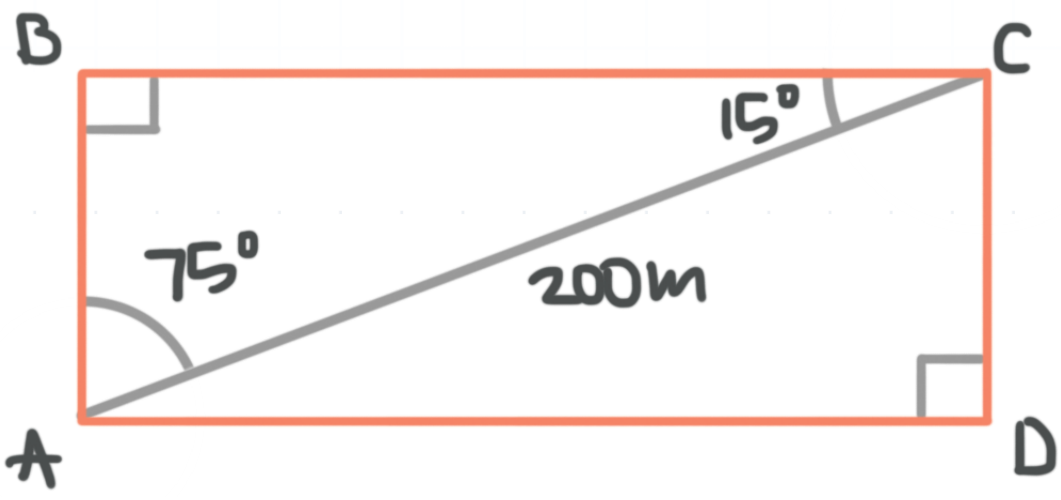
■ 3. A rectangular plot of land has a diagonal that's 200 meters long and creates interior angles 15° and 75° on the same side of the diagonal. If the owner wants to build a fence around the entire plot, find the length of the fence.



Solution:

If we call the rectangle $ABCD$, then we need to find the perimeter of $ABCD$.



Because the diagonal creates two right triangles, we can write two cosine equations.

$$\cos BAC = \frac{AB}{AC}$$

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$$\cos ACB = \frac{BC}{AC}$$

These become

$$AB = AC \cdot \cos BAC$$

$$BC = AC \cdot \cos ACB$$

Then we can plug in what we were given.

$$AB = 200 \cos 75^\circ$$

$$BC = 200 \cos 15^\circ$$

Because opposite sides of a rectangle have equal length, we can write the equation for perimeter as

$$P_{ABCD} = AB + BC + CD + AD$$

$$P_{ABCD} = 2AB + 2BC$$

$$P_{ABCD} = 2 \cdot 200 \cos 75^\circ + 2 \cdot 200 \cos 15^\circ$$

$$P_{ABCD} = 400(\cos 75^\circ + \cos 15^\circ)$$

Apply the sum-to-product identity

$$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$

to get

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$$P_{ABCD} = 400\left(2\cos\frac{75^\circ + 15^\circ}{2}\cos\frac{75^\circ - 15^\circ}{2}\right)$$

$$P_{ABCD} = 400\left(2\cos\frac{90^\circ}{2}\cos\frac{60^\circ}{2}\right)$$

$$P_{ABCD} = 400(2\cos 45^\circ \cos 30^\circ)$$

$$P_{ABCD} = 400 \cdot 2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$P_{ABCD} = 200\sqrt{2}\sqrt{3}$$

The perimeter of the rectangular plot, and the length of the fence, is

$$P_{ABCD} = 200\sqrt{6} \text{ meters}$$

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USING TRIG IDENTITIES TO PROVE THE EQUATION

■ 1. Prove the trig identity.

$$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$$

Solution:

Start by working on the numerator $1 - \cos x$, using a half-angle identity.

$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$$

$$2\sin^2\frac{x}{2} = 1-\cos x$$

$$1-\cos x = 2\sin^2\frac{x}{2}$$

We'll make this substitution later. Then work on the denominator $\sin x$, using the double-angle identity

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Make a substitution of $\theta = x/2$.

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

Then swap out the values from the original equation with the ones we just found.

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$$\tan\frac{x}{2}=\frac{1-\cos x}{\sin x}$$

$$\tan\frac{x}{2}=\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$$

$$\tan\frac{x}{2}=\tan\frac{x}{2}$$

2. Prove the trigonometric identity.

$$\frac{\sin 5x-\sin x}{\cos 5x+\cos x}=\tan 2x$$

Solution:

Let's try to simplify the left side of the equation to show that it's equivalent to the right side.

Apply the difference-to-product identity

$$\sin\theta-\sin\phi=2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

to the numerator from the left side.

$$\sin 5x-\sin x$$

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$$2\cos\frac{5x+x}{2}\sin\frac{5x-x}{2}$$

$$2\cos\frac{6x}{2}\cdot\sin\frac{4x}{2}$$

$$2\cos 3x\cdot\sin 2x$$

We'll make this substitution later. Apply the following sum-to-product identity

$$\cos\theta+\cos\phi=2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$

to the denominator.

$$\cos 5x+\cos x$$

$$2\cos\frac{5x+x}{2}\cos\frac{5x-x}{2}$$

$$2\cos\frac{6x}{2}\cdot\cos\frac{4x}{2}$$

$$2\cos 3x\cdot\cos 2x$$

Then swap out the values from the original equation with the ones we just found.

$$\frac{\sin 5x-\sin x}{\cos 5x+\cos x}=\tan 2x$$

$$\frac{2\cos 3x\cdot\sin 2x}{2\cos 3x\cdot\cos 2x}=\tan 2x$$

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$$\frac{\sin 2x}{\cos 2x} = \tan 2x$$

$$\tan 2x = \tan 2x$$

3. Prove the trigonometric identity.

$$\sin(x - \pi)\sin(x + \pi) = \sin^2 x$$

Solution:

Apply the product-to-sum identity

$$\sin\theta \sin\phi = \frac{1}{2}\big(\cos(\theta - \phi) - \cos(\theta + \phi)\big)$$

to the left side of the equation.

$$\sin(x - \pi)\sin(x + \pi) = \sin^2 x$$

$$\frac{1}{2}\Big(\cos\left[x - \pi - (x + \pi)\right] - \cos\left[x - \pi + x + \pi\right]\Big) = \sin^2 x$$

$$\frac{1}{2}\big(\cos(-2\pi) - \cos(2x)\big) = \sin^2 x$$

$$\frac{1}{2}\big(1 - \cos(2x)\big) = \sin^2 x$$

Apply the half-angle identity

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$$\sin^2\frac{\theta}{2}=\frac{1-\cos\theta}{2}$$

Substitute $x=\theta/2$ to get

$$\sin^2 x=\frac{1-\cos 2x}{2}$$

Substitute this value into the right side of the equation.

$$\frac{1}{2}\left(1-\cos(2x)\right)=\sin^2 x$$

$$\frac{1}{2}\left(1-\cos(2x)\right)=\frac{1-\cos 2x}{2}$$

$$\frac{\left(1-\cos(2x)\right)}{2}=\frac{1-\cos 2x}{2}$$

$$\frac{1-\cos 2x}{2}=\frac{1-\cos 2x}{2}$$

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LISTING THE SOLUTIONS IN THE INTERVAL

■ 1. Maggie rides a ferris wheel at a carnival. The altitude $h(t)$ of Maggie's cabin above the ground is modeled by $h(t)$ when $t \geq 0$ where t is time in minutes. Find all values of t when Maggie's cabin is at the bottom of the wheel.

$$h(t) = 16 - 15 \sin\left(\pi t + \frac{\pi}{2}\right)$$

Solution:

Maggie's cabin is at the bottom of the wheel when the function $h(t)$ is at its minimum value. Since $h(t)$ has an amplitude of 15 and range of $[-15,15]$, $h(t)$ reaches its minimum when

$$-15 \sin\left(\pi t + \frac{\pi}{2}\right) = -15$$

$$\sin\left(\pi t + \frac{\pi}{2}\right) = 1$$

The equation $\sin\theta = 1$ is true at all multiples of $\pi/2$.

$$\theta = \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 4\pi, \ldots$$

$\theta = \frac{\pi}{2} + 2\pi n$ where n is any integer

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Substitute $\theta = \pi t + \pi/2$ into this expression.

$$\pi t + \frac{\pi}{2} = \frac{\pi}{2} + 2\pi n$$

$$\pi t = 2\pi n$$

$$t = 2n$$

Since $t \geq 0$, n is any nonnegative integer. So Maggie is at the bottom of the wheel at $t = 0, 2, 4, 6, \ldots$ minutes from the start. So the solution in set notation is

$$\{2n \mid n \geq 0 : \ n \in \mathbb{Z}\}$$

■ 2. A ball is bouncing on a flat surface. Within the time interval $t \in [0,4]$, the height of each bounce is equal, and the altitude $h(t)$ of the ball's position above the surface is given by $h(t)$, when $0 \leq t \leq 4$ where t is time in seconds. Find all values of t when the ball reaches its highest altitude.

$$h(t) = 3 + 3\cos(3t)$$

Solution:

The ball will have its highest altitude whenever $h(t)$ reaches its maximum value. Since $3\cos(3t)$ has an amplitude of 3 and range of $[-3,3]$, $h(t)$ reaches its maximum when

$$3\cos(3t) = 3$$

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$$\cos(3t) = 1$$

From the unit circle we know that $\cos\theta = 1$ is true at all multiples of 2π.

$$\theta = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ \ldots$$

$\theta = 2\pi n$ where n is any integer

Substitute $\theta = 3t$ into this expression above.

$$3t = 2\pi n$$

$$t = \frac{2}{3}\pi n$$

Since $0 \le t \le 4$, we need to choose the solutions that lie in the interval $[0,4]$, which are only these:

$n = 0$ $\qquad$ $t = 0$

$n = 1$ $\qquad$ $t = \frac{2\pi}{3} \approx 2.1$

3. Solve the equation.

$$\cos^2 x - 3\cos x + 2 = 0$$

Solution:

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The equation is a quadratic equation in terms of $\cos x$. Substitute $y = \cos x$ to get

$$y^2 - 3y + 2 = 0$$

$$(y - 1)(y - 2) = 0$$

$y_1 = 1$ and $y_2 = 2$

So there are two possible solutions.

$$\cos x_1 = 1$$

$$\cos x_2 = 2$$

The second solution $\cos x_2 = 2$ has no solutions since the cosine function has the range $[-1,1]$ and can't be equal to 2. So let's solve the first equation.

From the unit circle we know that $\cos\theta = 1$ is true at all multiples of 2π.

$$x_1 = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ \ldots$$

$x_1 = 2\pi n$ where n is any integer

So the solution in set notation is

$$\{2\pi n :\ \ n \in \mathbb{Z}\}$$

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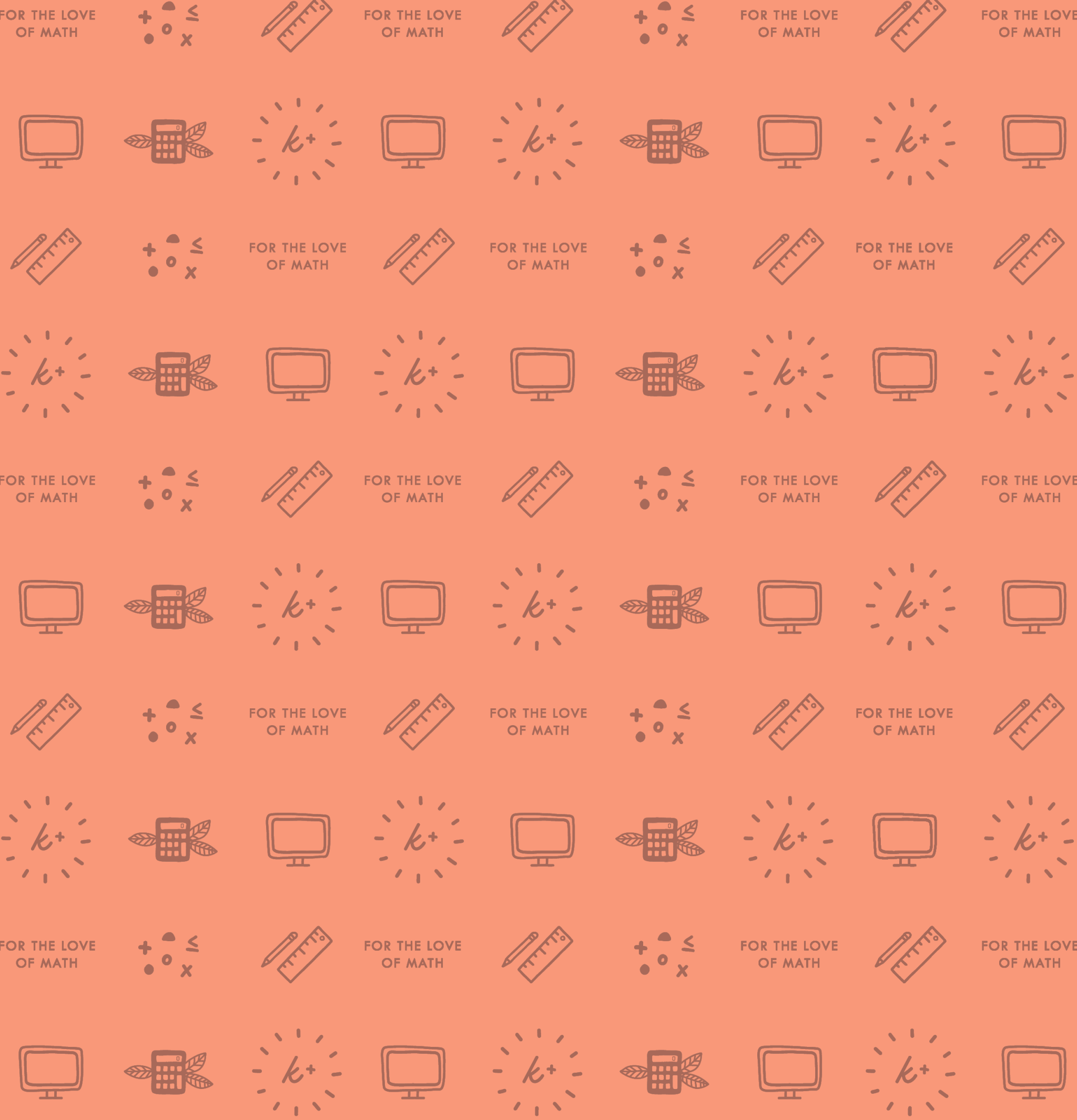

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